Semisimple centralisers are semi-direct products

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The sequence $(*)$ always splits, that is there is a finite subgroup $A_0 \subset C_{\mathbf{G}}(s)$ such that $C_{\mathbf{G}}(s)$ is a semi-direct product $C_{\mathbf{G}}(s)^0 \rtimes A_0$.

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- $C_{\mathbf{G}}(s)^0 = \langle \mathbf{T}, \{\mathbf{U}_\alpha \mid \alpha \in \Phi(s)\}\rangle$ *where* $\Phi(s) = \{\alpha \in \Phi \mid \alpha(s) = 1\}$ *is a reductive group with root system* $\Phi(s)$ *and Weyl group* $W^0(s) = \{s_\alpha \mid \alpha \in \Phi(s)\}$.

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If we can find a group morphism $A_W(s) \to N_{C_{\mathbf{G}}(s)}(\mathbf{T})$ we can split (**) and thus (*).

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We denote by $\Phi^\vee \subset Y(\mathbf{T})$ the coroots of $\mathbf{G}$ with respect to $\mathbf{T}$ and by $\alpha \mapsto \alpha^\vee$ the natural bijection $\Phi \xrightarrow{\sim} \Phi^\vee$.

Tits liftings and reduction to characteristic $\neq 2$

We choose a Borel subgroup $\mathbf{T} \subset \mathbf{B}$, which defines a positive root system Φ^+ and its basis Π, and a Coxeter system (W, S) where $S = \{s_\alpha \mid \alpha \in \Pi\}$.

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The second item implies that if k is of characteristic 2, the map σ is a group morphism $W \xrightarrow{\sigma} N_{\mathbf{G}}(\mathbf{T})$, in particular $A_W(s) \xrightarrow{\sigma} N_{\mathbf{G}}(\mathbf{T})$ solves the problem. From now on we assume the characteristic $\neq 2$.

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(generalize Adams-Vogan)

Let $\mathbf{b} \in B$ be a lift of w and let $\tilde{\mathbf{b}}$ be the reversed word of $\mathbf{b}$ (a lift of w^{-1}). Then $\boldsymbol{\sigma}(\mathbf{b}\tilde{\mathbf{b}}) = (\rho^\vee - w(\rho^\vee))/2$ where $\rho^\vee = (\sum_{\alpha \in \Phi^+} \alpha)/2$.

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The proof uses that there exists $z \in (Z\mathbf{G})^0$ such that $s' = sz \in \mathbf{G}'$.

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The element s is identified with an element of $Y(\mathbf{T}) \otimes (\mathbb{Q}/\mathbb{Z})_{p'} = (Y(\mathbf{T}) \otimes \mathbb{Q})_{p'}/Y(\mathbf{T})$. Let λ be a representative of s in $Y(\mathbf{T}) \otimes \mathbb{Q}$.
Let $\mathbf{T}_{\mathrm{sc}} = \pi^{-1}(\mathbf{T})$. Then $Y(\mathbf{T}_{\mathrm{sc}})$ is the root lattice and $Y(\mathbf{T}_{\mathrm{sc}}) \rtimes W$ is the affine Weyl group.

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We would like to reduce to the case of a quasi-simple group. In general semisimple groups are not product of their quasi-simple components, but this happens for adjoint or simply connected groups. We will use comparison with the simply connected covering $\mathbf{G}_{\mathrm{sc}}$ of $\mathbf{G}$, then decompose $\mathbf{G}_{\mathrm{sc}}$ in components.

We have an exact sequence $1 \to Z \to \mathbf{G}_{\mathrm{sc}} \xrightarrow{\pi} \mathbf{G} \to 1$ where Z is central in $\mathbf{G}_{\mathrm{sc}}$. We will find that we can identify $A_{\mathbf{G}}(s)$ to a subgroup of Z.

The element s is identified with an element of $Y(\mathbf{T}) \otimes (\mathbb{Q}/\mathbb{Z})_{p'} = (Y(\mathbf{T}) \otimes \mathbb{Q})_{p'}/Y(\mathbf{T})$. Let λ be a representative of s in $Y(\mathbf{T}) \otimes \mathbb{Q}$.

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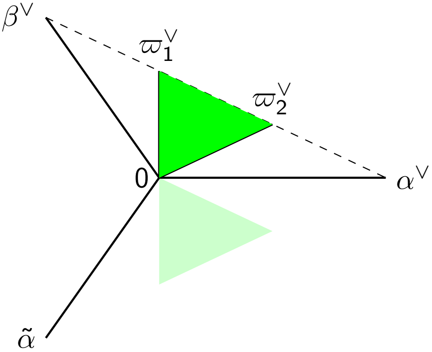
We will show that we can actually lift $\mathcal{A}_{\mathbf{G}}$ to $N_{\mathbf{G}}(\mathbf{T})$ by a group morphism, which solves the problem.

Type A_2



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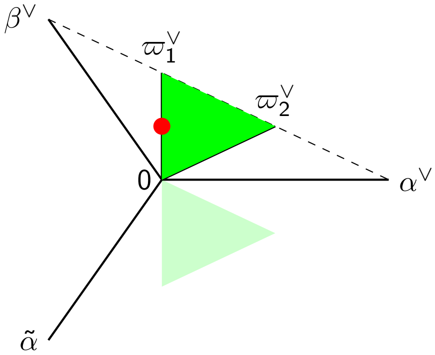

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The red dot represents a semisimple element stable by s_α; here $W(s) = W^0(s) = \langle s_\alpha \rangle$.

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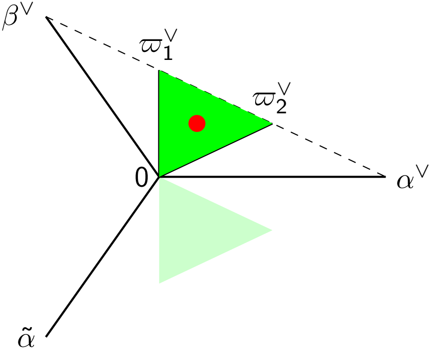

$\mathcal{A} = \langle s_\alpha s_\beta \rangle$: it preserves $\Pi^\vee \cup \{\tilde\alpha\} = \{\alpha^\vee, \beta^\vee, \tilde\alpha\}$ and rotates the picture. The red dot represents $diag(1, \zeta_3, \zeta_3^2)$ in PGL_3. Here $W^0(s) = 1$ and $W(s) = \langle s_\alpha s_\beta \rangle$.

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(definition)

We say that a lift $\tau(c) \in N_{\mathbf{G}_{\mathrm{sc}}}(\mathbf{T}_{\mathrm{sc}})$ of $c \in \mathcal{A}$ has property $(\flat)$ if

$$\tau(c)^{o(c)} = \begin{cases} \iota(c^{o(c)/2}) & \text{if } o(c) \text{ is even,} \\ 1 & \text{otherwise.} \end{cases} \tag{$\flat$}$$

The proof

We finish by the following 3 propositions.

If there is a lift τ of $\mathcal{A}$ to commuting elements of $N_{\mathbf{G}_{\mathrm{sc}}}(\mathbf{T}_{\mathrm{sc}})$ such that for all $a \in \mathcal{A}$, the element $\tau(a)$ has property $(\flat)$, then we can find a group lift of $\mathcal{A}_{\mathbf{G}}$ to $N_{\mathbf{G}}(\mathbf{T})$.

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This last one requires computations in each case, similar to what we did in type A. The braid group is useful.

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Let $\mathbf{T}$ *be a maximal torus of* $C_{\mathbf{G}}(s)^0$ *in an* F*-stable Borel subgroup; the following sequence is exact*

$$1 \to N_{C_{\mathbf{G}}(s)^0}(\mathbf{T})^F \to N_{C_{\mathbf{G}}(s)}(\mathbf{T})^F \to A_{\mathbf{G}}(s)^F \to 1. \qquad (**)$$

Finite reductive groups

We assume now that $k = \overline{\mathbb{F}_q}$ and F is a Frobenius endomorphism attached to an $\mathbb{F}_q$-structure on $\mathbf{G}$.

Let s be a semisimple element such that $C_{\mathbf{G}}(s)$ is F-stable (which is the case in particular if $s \in \mathbf{G}^F$; however our more general assumption turns out to be more suited to a reduction argument); the sequence

$$1 \to C_{\mathbf{G}}(s)^{0F} \to C_{\mathbf{G}}(s)^F \to A_{\mathbf{G}}(s)^F \to 1 \qquad (*)$$

is exact, since in the Galois cohomology exact sequence the next term $H^1(F, C_{\mathbf{G}}(s)^0)$ is trivial by Lang's theorem.

Let $\mathbf{T}$ be a maximal torus of $C_{\mathbf{G}}(s)^0$ in an F-stable Borel subgroup; the following sequence is exact

$$1 \to N_{C_{\mathbf{G}}(s)^0}(\mathbf{T})^F \to N_{C_{\mathbf{G}}(s)}(\mathbf{T})^F \to A_{\mathbf{G}}(s)^F \to 1. \qquad (**)$$

Such tori are conjugate under $(C_{\mathbf{G}}(s)^0)^F$.

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Let $a \in W$ be of minimal length in its W-conjugacy class and let $\mathbf{a} = \tilde{\pi}(a)$ be the lift of a to B. Then the natural map $C_B(\mathbf{a}) \to C_W(a)$ is surjective.

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For types $D_{2n}, {}^2D_{2n}, {}^3D_4$ we have to do more work.